General interest-rate models and the universality of HJM

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There are now many models of interest-rate markets available. Many are based on the powerful HJM model of Heath, Jarrow and Morton (1992). Others, surveyed by Rogers (1995), are not (explicitly) set within the HJM framework, but are driven by, say, the short-term interest rate. In this paper we shall describe the appearance of a general interest-rate market. We shall also show, under some additional technical restrictions, that the general model is the short-rate model, and that the short-rate model is the HJM model. In other words, every 'sufficiently nice' model is simultaneously an HJM model and a short-rate model.

1. Introduction

We choose to work within the framework where all our processes are adapted to the filtration of an n-dimensional Brownian motion.

We do this for three important reasons. Firstly, there are great technical simplifications, such as the continuity of all martingales, which allow stronger results. Secondly, the Brownian case is recognizably distinct from other frameworks such as Poisson processes and Markov chains, and worthy of consideration. And thirdly, much market practice and interest is focused in this direction.

But, this framework assumption aside, we will try to be as general as possible.

Many approaches to interest-rate modelling can be divided into two types: short rate (SR) models and forward rate (HJM) models. A market of discount bonds $P(t,T)$ can be defined as

(SR) $$P(t,T) = \mathbb{E}\Big(\exp\big(-\textstyle\int_t^T r_u\,du\big)\ \Big|\ \mathcal{F}_t\Big),$$

for some adapted short-rate process r_t,

(HJM) $$P(t,T) = \exp\big(-\textstyle\int_t^T f(t,u)\,du\big),$$

for some family of forward rate processes $f(t,T)$.

In each case, the defining equation is augmented by some conditions or constraints on the driving process(es). In SR, the process r_t must be such that the integral $\int_0^T |r_s|\,ds$ exists and that the expectation of the reciprocal of the bank account process $B_t = \exp(\int_0^t r_s\,ds)$ is finite. In HJM, it is firstly assumed that the bonds are differentiable in T to give the forward rates $f(t,T)$, and further that $f(t,T)$ is a continuous semimartingale in t whose volatility and drift satisfy certain conditions.

It is immediate that a HJM model is also a SR model. This is because if we take the short rate process r_t to be $f(t,t)$, then the SR-equation is just equation (19) of Heath, Jarrow and Morton (1992).

It is not so obvious that any SR model is also a HJM model. The bond prices might not even be T-differentiable, and the other HJM conditions are awkward to prove directly. We shall address this question and show that:

Theorem. *An SR model, satisfying the regularity condition* $\mathbb{E}\int_0^T |r_u| B_u^{-1}\, du < \infty$*, is also an HJM model.*

These models are both prescriptive of the interest-rate market — its form is determined by these definitions. It would be interesting to have a descriptive picture of the market, which would tell us whether there are any other models we could use. We will answer this too and show, in fact, that there are not. Before we can state the theorem, we need to say what a general interest-rate market adapted to the Brownian filtration looks like.

We will take as axioms that there is a market of discount bonds $P(t,T)$ such that for each maturity T,

(BM)
- $P(t,T)$ is positive for all $t \leqslant T$
- $P(T,T) = 1$
- there is no arbitrage in the market
- $P(t,T)$ is a continuous semimartingale in t.

We might call these the BM or bond market axioms, and it is immediate that all models of either SR or HJM type satisfy them. We shall further justify the axioms momentarily, but we can now state a theorem about such a market:

Theorem. *A model satisfying BM will have a short rate process* r_t *and, if the corresponding bank account process* B_t *is tradable or hedgeable, it is also an SR model.*

Some justification of the BM axioms is necessary to see that any useful model must have these characteristics. The positivity of $P(t,T)$ follows from the economic assumptions of the bond holder's limited liability, which keeps things non-negative, and of a utility function which is not completely indifferent to future rewards, which gives the strict positivity. Also the bond cannot default, which implies that $P(T,T)$ is exactly 1. The no-arbitrage condition speaks for itself.

If there is no arbitrage, work by Delbaen and Schachermayer (1994) proves that if a security is adapted (you can't see into the future), is right-continuous with left-limits (the right way round to avoid arbitrage profits from discontinuous shocks), and is locally bounded (doesn't explode) then firstly it is a semimartingale (their theorem 7.2) and secondly there is a measure under which it is a local martingale (their corollary 1.2). The security is then continuous, as are all Brownian local martingales. We can thus replace the continuous semimartingale condition by the equivalent requirement that

- $P(t,T)$ is locally bounded, adapted and right-continuous with left-limits.

Although there is a martingale measure for each bond individually, there is not yet a proof that there is a measure under which simultaneously all bonds are martingales (though work in preparation by Lowther will go a good way towards this). Until then, we shall assume that

- there is a measure which makes the (discounted) bond prices into martingales.

Ab initio, neither SR nor HJM models can claim to be truly general. The SR model assumes the existence of a short-rate process r_t, and the HJM model assumes the existence of the forward rates $f(t,T)$. We shall show that these assumptions are warranted and that any market of bonds satisfying BM, plus two regularity conditions, is both an SR model and an HJM model. This includes the burden of showing that a short-rate process exists, and that the bond prices are jointly measurable and T-differentiable.

In the next Section we will lay out our main theorems. To show that a model is HJM we need to prove the joint measurability of some families of processes parameterised by maturity as well as a stochastic version of Fubini's theorem. The existing stochastic calculus literature does not seem to address our precise problem, and Section 3 contains the technical details of how they may be solved. The subsequent Section then uses those results to prove the original theorems.

Once we have understood the full generality of the interest-rate market it is possible to explore its wilder shores. Section 5 contains two examples of pathological behaviour of BM models. One is an example for which one of the regularity conditions fails, and the model is neither SR nor fully HJM. The other is a model which is both SR and HJM, some of whose bond prices have non-vanishing volatility as time approaches maturity.

2. Main results

As stated earlier, we shall show that any model adapted to a finite Brownian motion filtration and satisfying some technical constraints is both an SR and an HJM model. Our generality allows us to separate out properties of an interest-rate model which are absolutely essential for the mathematics to operate, and those which are just desirable for modelling or econometric purposes.

Notation note: to ease the proliferation of subscripts, it may be assumed that a summation sign without limits is being summed over the range 1 to n. For example, we will write $\sum_i \sigma_i \, dW_i$ for $\sum_{i=1}^n \sigma_i \, dW_i$.

For ease of proof, we shall also assume that all the semimartingales involved have absolutely continuous drifts.

We have three theorems.

Theorem 1. *Let $P(t,T)$ be a market of pure discount bonds under a measure $\mathbb{P}$, with the boundary condition that $P(T,T) = 1$ for every maturity date T. We assume that*

(BM1) for each maturity date T the process $P(t,T)$ is a positive-valued continuous semimartingale in t and is adapted to the filtration $\mathcal{F}_t$ of n-dimensional Brownian motion.

(BM2) the market is 'arbitrage-free', in the sense that for any particular bond there is a measure equivalent to $\mathbb{P}$ under which the bonds (discounted by the chosen bond) are martingales.

Then

(i) for each maturity date T, there exist $\mathcal{F}$-previsible processes $\Sigma_i(t,T)$ $(i=1,\ldots,n)$ and $\alpha(t,T)$ such that $\int_0^T (|\Sigma(t,T)|^2 + |\alpha(t,T)|)\,dt < \infty$ and

$$d_t P(t,T) = P(t,T)\Big(\sum_i \Sigma_i(t,T)\,dW_i(t) + \alpha(t,T)\,dt\Big).$$

(ii) fixing a maturity horizon date τ and choosing the bond $P(t,\tau)$ to be numeraire, there is a measure $\mathbb{P}^\tau$ equivalent to $\mathbb{P}$ and a $\mathcal{F}$-previsible n-vector $\gamma_i(t)$ $(i=1,\ldots,n)$ such that

$$\left.\frac{d\mathbb{P}^\tau}{d\mathbb{P}}\right|_{\mathcal{F}_t} = \exp\left(-\sum_i \int_0^t \gamma_i(s)\,dW_i(s) - \tfrac{1}{2}\int_0^t |\gamma(s)|^2\,ds\right),$$

and $W_i^\tau(t) = W_i(t) + \int_0^t \gamma_i(s)\,ds$ is $\mathbb{P}^\tau$-Brownian motion. Additionally, for every maturity date T

$$d_t P(t,T) = P(t,T)\Big(\sum_i \Sigma_i(t,T)\,dW_i^\tau(t) + \tilde{\alpha}(t,T)\,dt\Big),$$

where $\tilde{\alpha}(t,T) = \alpha(t,T) - \sum_i \gamma_i(t)\Sigma_i(t,T)$ is also a $\mathcal{F}$-previsible process whose integral $\int_0^T |\tilde{\alpha}(t,T)|\,dt$ is finite. The τ-bond discounted bond price

$$\frac{P(t,T)}{P(t,\tau)} \quad \text{is a } \mathbb{P}^\tau\text{-martingale for every } T.$$

(iii) there is a $\mathcal{F}$-previsible process r_t such that $\int_0^\tau |r_t|\,dt < \infty$ and

$$d_t P(t,T) = P(t,T)\left(\sum_i \Sigma_i(t,T)\,dW_i^\tau(t) + \Big(\sum_i \Sigma_i(t,T)\Sigma_i(t,\tau) + r_t\Big)\,dt\right).$$

Also the process $B_t = \exp\big(\int_0^t r_s\,ds\big)$ exists and is absolutely continuous.

(iv) there is a version of $P(t,T)$ which is jointly measurable and t-continuous, and such that

$$P(t,T) = P(t,\tau)\,\mathbb{E}_{\mathbb{P}^\tau}\big(P^{-1}(T,\tau)\;\big|\;\mathcal{F}_t\big).$$

Further there are jointly measurable versions of $\Sigma_i(t,T)$ and $\tilde{\alpha}(t,T)$.

It should be noted that while the chosen numeraire τ does provide a local time horizon for the market, it need not be an ultimate limit. The market can extend beyond τ, even up to infinity, but at or before τ a new numeraire must be picked to allow further progress.

Theorem 2. *Suppose that BM holds, then the market is a SR model if and only if the additional regularity condition holds; that*

$$\text{(A1)} \qquad \zeta_t = \exp\left(-\sum_i \int_0^t \Sigma_i(s,\tau)\,dW_i^\tau(s) - \tfrac{1}{2}\int_0^t |\Sigma(s,\tau)|^2\,ds\right)$$

is a uniformly integrable $\mathbb{P}^\tau$-martingale up to time τ (that is that $\mathbb{E}_{\mathbb{P}^\tau}(\zeta_\tau) = 1$, or it is sufficient that $\mathbb{E}_{\mathbb{P}^\tau} \exp \frac{1}{2} \int_0^\tau |\Sigma(t,\tau)|^2 \, dt < \infty$).

This condition is also equivalent to each of the following

(v) *the τ-bond discounted value of the process B_t, $P^{-1}(t,\tau)B_t$ is a $\mathbb{P}^\tau$-martingale.*

(vi) *there exists a measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ and $\mathbb{P}^\tau$ such that $\tilde{W}_i(t)$ defined to be $W_i^\tau(t) + \int_0^t \Sigma_i(s,\tau)\, ds$ is $\mathbb{Q}$-Brownian motion and*

$$d_t P(t,T) = P(t,T)\Big(\sum_i \Sigma_i(t,T)\, d\tilde{W}_i(t) + r_t \, dt\Big).$$

(vii) *there exists a measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that the B-discounted bond prices $B_t^{-1}P(t,T)$ are $\mathbb{Q}$-martingales and the bond prices themselves can be written as the expectation*

$$P(t,T) = \mathbb{E}_{\mathbb{Q}}\Big(\exp(-\textstyle\int_t^T r_u \, du) \;\Big|\; \mathcal{F}_t\Big).$$

Theorem 3. *Suppose that BM and (A1) all hold and further that*

(A2) *the expectation $\mathbb{E}_{\mathbb{Q}}(\int_0^\tau |r_u| B_u^{-1} \, du)$ is finite,*

then the market is an HJM model, in that

(viii) *the bond price $P(t,T)$ is absolutely continuous in T with $-\frac{\partial}{\partial T}P(t,t) = r_t$, and*

$$-\frac{\partial}{\partial T} \log P(t,T) = f(t,T) := \frac{\mathbb{E}_{\mathbb{Q}}\big(r_T \exp(-\int_t^T r_u \, du) \mid \mathcal{F}_t\big)}{P(t,T)}.$$

(ix) *the process $f(t,T)$ is a semimartingale in t with* SDE

$$d_t f(t,T) = \sum_i \sigma_i(t,T)\, d\tilde{W}_i(t) - \sum_i \sigma_i(t,T)\Sigma_i(t,T)\, dt,$$

where $\sigma_i(t,T)$ $(i = 1, \ldots, n)$ is a $\mathcal{F}$-previsible process in t with $\int_0^T |\sigma(t,T)|^2 \, dt < \infty$.

(x) *the bond volatilities $\Sigma_i(t,T)$ are absolutely continuous in T with $\Sigma_i(t,t) = 0$ and*

$$\Sigma_i(t,T) = -\int_t^T \sigma_i(t,u)\, du.$$

Theorem 1 describes what we say about the basic BM model with no additional regularity conditions. The conditions (BM1) and (BM2) are our formulation of the BM axioms. Parts (i) and (ii) recall familiar material. There is a link between a market being arbitrage-free and the existence of an equivalent measure, under which the discounted securities are martingales. Quite what this link is in the case of an infinite number of tradable securities is an open question, but the finite case has been explored widely from Harrison and Pliska (1981) to Delbaen and Schachermayer (1994). Here the market has a risk-neutral τ-forward measure, under which bond prices (discounted by the numeraire τ-bond) are martingales.

The theorem continues by showing that bond drift is a function only of the volatilities and a process (suggestively) labelled r_t. If the model is SR then r_t will be the short rate, but we do not know this for sure yet. At this stage the putative bank account process (or cash bond) B_t exists as a mathematical process, but may not be a tradable security. The burden of result (iv) is less the formula for $P(t,T)$ in terms of the τ-bond, than in the statements of the joint measurability of P, Σ_i and $\tilde{\alpha}$. These (or equivalent) measurabilities were assumed by HJM, but we have them as results.

Theorem 2 contains nothing new technically, but is included to make clear the role of r_t and the bond B_t. Mathematically, Theorem 1 posits the existence of the bonds $P(t,T)$ and doesn't specify the existence or tradability of B_t. In fact we showed in part (iii) that B_t did exist, but that didn't prove it was tradable. By the equivalence of lack of arbitrage and existence of a martingale measure, the bond B_t can be traded only if it is a martingale under the forward measure $\mathbb{P}^\tau$ (when discounted). As stated, the regularity condition (A1) is actually equivalent to results (v), (vi), (vii) and the tradability of B_t without arbitrage. Although almost all models do assume this condition, it needn't happen and a completely general model need not satsify condition (A1). An example of a market where the cash bond is not a $\mathbb{P}^\tau$-martingale is given in Section five. If the condition does hold, then the bonds themselves can now be expressed as expectations of discount factors involving r_t, and the market is seen to be an SR model. The bonds discounted by the cash bond are $\mathbb{Q}$-martingales, but the new measure $\mathbb{Q}$ may depend on the τ originally chosen if the market is incomplete.

The main results of Theorem 3 are (viii) and (x) which prove that $P(t,T)$ and $\Sigma_i(t,T)$ respectively are absolutely continuous in T. In their paper, HJM assume the differentiability of $P(t,T)$ and that the resulting forward rates are semimartingales. Here we show that those assumptions are unnecessary. The condition (A2) makes the bond term-structure differentiable (at almost all maturities) and ensures that the forward rate process is well-behaved. It is this that actually lets us see the cash bond as the limit of holding very short-dated bonds and makes it tradable. What we have by the end of item (x) is an HJM model, as defined by Heath, Jarrow and Morton (1992). It should be noted that any model of the bond market without continuous yield curves, differentiable at almost all maturities, cannot be satisfying assumptions BM and (A1–2). We also note that it is sufficient that the interest rate r_t be bounded below by some constant, that is $r_t \geqslant -K$ for all $t \leqslant \tau$, for condition (A2) to hold.

We can now address the first theorem we stated in Section 1.

Corollary 4. *Suppose that r_t is a previsible process adapted to the Brownian motion filtered space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\int_0^\tau |r_t|\,dt$ is finite and that $\mathbb{E}\int_0^\tau |r_u| B_u^{-1}\,du$ is also finite, where $B_t = \exp(\int_0^t r_s\,ds)$. Then the market of bond prices $P(t,T)$, defined by*

$$P(t,T) = \mathbb{E}\Big(\exp\big(-\textstyle\int_t^T r_u\,du\big) \;\Big|\; \mathcal{F}_t\Big),$$

is an arbitrage-free market satisfying the conditions BM and (A1–2) of the theorems and results (i) to (x).

In other words, an SR model satisfying (A2) is an HJM model.

3. Preliminary results

To prove the three theorems, we do require some technical results on the measurability and existence of some random processes, as well as a stochastic variant of Fubini's theorem. In all of what follows, we work with a one-dimensional Brownian motion W_t, its filtration $\mathcal{F}_t$, and its probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The one-factor case is presented purely for simplicity, and the obvious multi-factor versions of these results also hold.

Lemma 5. *If $(A, \mathcal{A})$ is a measurable space and $X : A \to L^1(\Omega, \mathcal{F}_\tau)$, $a \mapsto X_a$, is a measurable function (giving L^1 the Borel σ-algebra induced by its norm), then there is a jointly measurable function F*

$$F : A \times \Omega \to \mathbb{R},$$

such that $F(a, \omega)$ is a version of X_a for every a.

Proof of Lemma. We recall firstly that $L^1(\Omega, \mathcal{F}_\tau)$ is separable, because $L^2(\Omega, \mathcal{F}_\tau)$ is both separable itself (the filtration $\mathcal{F}_\tau$ has a countable basis, such as $\{W_q \leqslant q'\}$, $q \in \mathbb{Q} \cap [0, \tau]$, $q' \in \mathbb{Q}$) and is also a dense subspace of $L^1(\Omega, \mathcal{F}_\tau)$. Let (Y_n) denote a dense sequence in L^1, choosing a version $Y_n(\omega)$ of each one.

Then for any positive ϵ, we define the measurable index $n_\epsilon : L^1(\Omega, \mathcal{F}_\tau) \to \mathbb{N}$ by

$$n_\epsilon(X) = \inf\{n : \|X - Y_n\|_1 < \epsilon\}.$$

This lets us define an approximation to F as

$$F_\epsilon(a, \omega) = Y_{n_\epsilon(X_a)}(\omega),$$

which is certainly jointly measurable for every ϵ. For any a, there is a version $F_0(a, \omega)$ of X_a, which is ω-measurable, but F_0 is not necessarily a-measurable.

We can now use Markov's inequality to see that the set

$$A_\epsilon^a = \left\{\omega : |F_\epsilon(a, \omega) - F_0(a, \omega)| \geqslant \sqrt{\epsilon}\right\}$$

has size $\mathbb{P}(A_\epsilon^a) < \sqrt{\epsilon}$. Moving along the fast subsequence $\epsilon_n = 2^{-2n}$, we have that the set $A_0^a = \limsup_{n \to \infty} A_{\epsilon_n}^a$ is $\mathbb{P}$-null. For all ω not in A_0^a, $F_{\epsilon_n}(a, \omega)$ tends to $F_0(a, \omega)$. If we define

$$F(a, \omega) = \begin{cases} \lim_{n \to \infty} F_{\epsilon_n}(a, \omega) & \text{if this limit exists,} \\ 0 & \text{otherwise.} \end{cases}$$

Then $F(a, \omega)$ is jointly measurable, and is a version of X_a for every a. □

Proposition 6. *If $X : [0, \tau] \to L^1(\Omega, \mathcal{F}_\tau)$, $T \mapsto X_T$, is a measurable function, then there is a jointly measurable function*

$$N : [0, \tau] \times [0, \tau] \times \Omega \to \mathbb{R},$$

such that $N(t,T,\omega)$ is a t-continuous version of the martingale

$$N(t,T) = \mathbb{E}\big(X_T \mid \mathcal{F}_t\big).$$

Proof of Proposition. The function

$$E : [0,\tau] \times L^1(\Omega, \mathcal{F}_\tau) \to L^1(\Omega, \mathcal{F}_\tau),$$

which takes (t, X) to $\mathbb{E}(X|\mathcal{F}_t)$ is continuous. This is because

$$\|E(t,X) - E(s,Y)\|_1 \leqslant \|X - Y\|_1 + \|E(t,X) - E(s,X)\|_1,$$

and the second term of the right-hand side tends to zero as s tends to t for any X because the process $E(t,X)$ is uniformly integrable and (almost surely) continuous in t. Thus the continuous function E is jointly measurable, and so too must be the function $\tilde{X}$,

$$\tilde{X} : [0,\tau] \times [0,\tau] \to L^1(\Omega, \mathcal{F}_\tau),$$

which takes (t,T) to $E(t, X_T)$. The Lemma 5 now applies to give a jointly measurable (but not necessarily t-continuous) function $\tilde{N}(t,T,\omega)$ which is a version $\mathbb{E}(X_T|\mathcal{F}_t)$.

Following the notation of II.61 of Rogers and Williams (1994), the set A defined to be the set

$$\big\{(\omega,T) : \forall\epsilon > 0,\ \exists\delta > 0,\ \forall q, q' \in \mathbb{Q} \cap [0,\tau],\ |q - q'| < \delta \Rightarrow |\tilde{N}(q,T,\omega) - \tilde{N}(q',T,\omega)| < \epsilon\big\}$$

is measurable and has sections $A^T = \{\omega : (\omega, T) \in A\}$ of size $\mathbb{P}(A^T) = 1$ for every T. If we define

$$N(t,T,\omega) = \begin{cases} \lim_{q\downarrow\downarrow t} \tilde{N}(q,T,\omega) & \text{if } (\omega,T) \in A, \\ 0 & \text{if } (\omega,T) \notin A, \end{cases}$$

then this is a measurable t-continuous modification of $\tilde{N}$. □

Lemma 7. *Let $H^0 = H^0(\Omega, \mathcal{F}_\tau, \mathbb{P})$ be the space of $\mathcal{F}$-previsible processes ψ up to time τ such that $\int_0^\tau \psi_t^2\, dt$ is finite almost surely, and give H^0 the (metric) topology under which ψ_n is defined to converge to ψ if*

$$\int_0^\tau \big(\psi_n(t) - \psi_t\big)^2\, dt \to 0 \quad \textit{in probability.}$$

Define the map $\Phi : L^1(\Omega, \mathcal{F}_\tau, \mathbb{P}) \to H^0(\Omega, \mathcal{F}_\tau, \mathbb{P})$ which takes the random variable X to the process $\Phi_t(X)$ which is the Brownian motion representation of $\mathbb{E}(X|\mathcal{F}_t)$, that is

$$\mathbb{E}(X|\mathcal{F}_t) = \mathbb{E}(X) + \int_0^t \Phi_s(X)\, dW_s.$$

Then the map Φ is continuous.

Proof of Lemma. Let M_t^X be the martingale $\mathbb{E}(X|\mathcal{F}_t)$. By Doob's submartingale inequality, the maximum process $M_*^X = \sup_{t\leqslant\tau} |M_t^X|$ satisfies

$$\mathbb{P}(M_*^X \geqslant a) \leqslant \frac{\|X\|_1}{a},$$

and thus $M_*^X \to 0$ in probability as X tends to 0 in L^1. Define F to be the moderate (and bounded) function $F(x) = x/(x+1)$, so that the Burkholder-Davis-Gundy inequality (see IV.42 of Rogers and Williams (1987)) holds:

$$\mathbb{E}F\big([M^X]_\tau^{\frac{1}{2}}\big) \leqslant C_F\, \mathbb{E}F\big(M_*^X\big),$$

for some constant C_F. Now $F\big(M_*^X\big)$ tends to zero in probability and is bounded by 1, so converges in L^1. So $F\big([M^X]_\tau^{\frac{1}{2}}\big)$ converges in L^1 and hence in probability. As F is continuous, we deduce that

$$[M^X]_\tau \to 0 \quad \text{in probability,}$$

which is the desired result. □

Proposition 8. *Under the conditions of Proposition 6, there is a jointly measurable function*

$$\phi : [0,\tau] \times [0,\tau] \times \Omega \to \mathbb{R}$$

such that $\phi(t, T, \omega)$ is a version of the $\mathcal{F}$-previsible process $\Phi_t(X_T)$, and that

$$N(t,T) = N(0,T) + \int_0^t \phi(u,T)\, dW_u.$$

Proof of Proposition. Let $H^0(\Omega, \mathcal{F}_\tau, \mathbb{P})$ be as in Lemma 7. Recall our dense sequence Y_n in $L^1(\Omega, \mathcal{F}_\tau, \mathbb{P})$ and let $\psi_n(t,\omega)$ be a version of the process $\Phi_t(Y_n)$. Then for every positive ϵ the function $n_\epsilon : [0,\tau] \to \mathbb{N}$, definied by

$$n_\epsilon(T) = \min\{n : \|X_T - Y_n\| < \epsilon\}$$

is measurable. For each T, let $\phi_0(t,T)$ be the $\mathcal{F}$-previsible process $\Phi_t(X_T)$, which will not necessarily be T-measurable. Define our measurable approximation to ϕ as ϕ_ϵ, where

$$\phi_\epsilon(t,T,\omega) = \psi_{n_\epsilon(T)}(t,\omega).$$

For any T, the process $\phi_\epsilon(t,T)$ is just a version of $\Phi_t(Y_{n_\epsilon(X_T)})$ which tends in the H^0-topology to $\Phi_t(X_T)$ because Φ is continuous (by Lemma 7). As in Lemma 5, along a fast subsequence ϵ_n there will be almost sure convergence and we can define ϕ to be the limit

$$\phi(t,T,\omega) = \begin{cases} \lim_{n\to\infty} \phi_{\epsilon_n}(t,T,\omega) & \text{if this limit exists,} \\ 0 & \text{otherwise.} \end{cases}$$

This completes the proof. □

Proposition 9 (Stochastic Fubini). *If $X : [0,\tau] \to L^1(\Omega, \mathcal{F}_\tau)$, $u \mapsto X_u$, is a measurable function such that $\int_0^\tau |X_u|\,du$ is also in $L^1(\Omega, \mathcal{F}_\tau)$, then firstly conditional expectation commutes with integration in that*

$$\mathbb{E}\Big(\textstyle\int_0^\tau X_u\,du \;\Big|\; \mathcal{F}_t\Big) = \displaystyle\int_0^\tau \mathbb{E}(X_u|\mathcal{F}_t)\,du,$$

and secondly the map Φ of Lemma 7 also commutes with integration:

$$\Phi_t\Big(\textstyle\int_0^\tau X_u\,du\Big) = \displaystyle\int_0^\tau \Phi_t(X_u)\,du.$$

In other words, writing $\phi(t,u)$ for a measurable version of $\Phi_t(X_u)$, and Y for the integral $\int_0^\tau X_u\,du$, then

$$\mathbb{E}(Y|\mathcal{F}_t) - \mathbb{E}(Y) = \int_0^t \left(\int_0^\tau \phi(s,u)\,du\right)\,dW_s = \int_0^\tau \left(\int_0^t \phi(s,u)\,dW_s\right)\,du.$$

Proof of Proposition. Let us set $f(u)$ to be $\mathbb{E}(|X_u|)$. We know that $\int_0^\tau f(u)\,du$ is finite, so $f(u)$ must be finite for almost all times u. We can set X_u to be zero on the set of 'bad' u without loss of generality, and thus assume that $f(u)$ is finite for all times u. By Proposition 6, there is a jointly measurable function $N(t,u)$, which is a version of the martingale

$$N(t,u) = \mathbb{E}(X_u|\mathcal{F}_t).$$

We want to show that

$$\int_0^\tau N(t,u)\,du = \mathbb{E}\Big(\textstyle\int_0^\tau X_u\,du \;\Big|\; \mathcal{F}_t\Big).$$

The left-hand side above is $\mathcal{F}_t$-measurable and L^1-integrable. For any event A in $\mathcal{F}_t$

$$\mathbb{E}\Big(I_A \textstyle\int_0^\tau N(t,u)\,du\Big) = \displaystyle\int_0^\tau \mathbb{E}\big(I_A N(t,u)\big)\,du = \int_0^\tau \mathbb{E}\big(I_A X_u\big)\,du.$$

The standard version of Fubini's theorem allows us to rewrite this as

$$\mathbb{E}\left(I_A \textstyle\int_0^\tau X_u\,du\right).$$

which proves the first part of the result.

For the second part, we can define the stopping times

$$T_u^n = \inf\{t : |N(t,u)| \geqslant n\} \wedge \tau,$$

which are measurable in u. Then there is an approximation X^n to X given by

$$X_u^n = N(T_u^n, u),$$

which is jointly measurable and bounded. Because X_u^n is just X_u stopped at time T_u^n, it follows that

$$\Phi_t(X_u^n) = \Phi_t(X_u), \qquad t \leqslant T_u^n.$$

By the L^2-version of the stochastic Fubini theorem in Ikeda and Watanabe (1981), we can deduce that

$$\Phi_t\left(\int_0^\tau X_u^n \, du\right) = \int_0^\tau \Phi_t(X_u^n)\, du.$$

In addition, for each u the random variable X_u^n tends to X_u in $L^1(\Omega, \mathcal{F}_\tau)$ as n tends to infinity, because $\|X_u^n - X_u\|_1 \leqslant 2\mathbb{E}(|X_u|; T_u^n < \tau)$. As this upper bound converges (to zero) monotonically, we can also see that

$$\int_0^\tau X_u^n \, du \to \int_0^\tau X_u \, du \quad \text{in } L^1(\Omega, \mathcal{F}_\tau).$$

By Lemma 7, Φ is continuous, so

$$\Phi\left(\int_0^\tau X_u^n \, du\right) \to \Phi\left(\int_0^\tau X_u \, du\right) \quad \text{in } H^0(\Omega, \mathcal{F}_\tau).$$

Let $\Psi_n(t) = \Phi_t(\int_0^\tau X_u^n \, du) = \int_0^\tau \Phi_t(X_u^n)\, du = \int_{A_n(t)} \Phi_t(X_u)\, du$, where $A_n(t)$ is the set $\{u \in [0,\tau] : t < T_u^n\}$. Because $A_n(t)$ tends upwards to the whole of the interval $[0,\tau]$, as n tends to infinity, so $\Psi_n(t)$ converges to $\int_0^\tau \Phi_t(X_u)\, du$. (We actually have dominated convergence here as $\int_0^\tau |\Phi_t(X_u)|\, du$ is finite, seen by considering the related system $\tilde{X}$ which has $\Phi_t(\tilde{X}_u) = |\Phi_t(X_u)|$ and automatic (monotone) convergence.) Hence $\int_0^\tau \Phi_t(X_u)\, du$ is also equal to $\Phi_t(\int_0^\tau X_u \, du)$. □

4. Proofs of the theorems

We can now use our preliminary results from Section three to prove the results stated as theorems in the Section two.

Proof of Theorem 1. Result (i). Because $P(t,T)$ is a semimartingale, the drift $\alpha(t,T)$ exists by our assumption that the drift is absolutely continuous. The volatilities $\Sigma_i(t,T)$ exist by the Brownian martingale representation theorem. See, for instance, IV.36.5 of Rogers and Williams (1987).

Result (ii) is a re-statement of assumption (BM2). The change of measure comes from the Cameron-Martin-Girsanov theorem. See IV.38.5(i) of Rogers and Williams (1987).

Result (iii). Using the SDE for $P(t,T)$ in part (ii) above, the SDE for $Z(t,T) = P(t,T)/P(t,\tau)$ is

$$d_t Z(t,T) = Z(t,T)\Big(\sum_i \big(\Sigma_i(t,T) - \Sigma_i(t,\tau)\big)\, dW_i^\tau(t) + \big(r(t,T) - r(t,\tau)\big)\, dt\Big),$$

where $r(t,T)$ is the $\mathcal{F}$-previsible process $\tilde{\alpha}(t,T)-\sum_i \Sigma_i(t,T)\Sigma_i(t,\tau)$, such that the integral $\int_0^T |r(t,T)|\,dt$ is finite. For this to be a $\mathbb{P}^\tau$-local martingale for every T, the drift term must be zero — in other words, every $r(t,T)$ must equal $r(t,\tau)$. Supressing its dependence on the fixed maturity τ, we shall call this common value r_t. In other words, to be arbitrage free the drift must have the form

$$\tilde{\alpha}(t,T) = \sum_i \Sigma_i(t,T)\Sigma_i(t,\tau) + r_t,$$

from which the stated SDE follows. (The process r_t will only be fully independent of τ if the market is complete, which property we are not concerned with here.)

Result (iv). The formula follows from the assumption (BM2) that the discounted bond prices are $\mathbb{P}^\tau$-martingales. Joint measurability is harder, and follows from Proposition 6 applied to the function $T \mapsto P^{-1}(T,\tau)$. To show that this function is measurable it is enough to remark that it is the monotone limit of the functions $T \mapsto \max\{P^{-1}(T,\tau), K\}$ as K goes to infinity, which itself is continuous (and hence measurable) as a function from $[0,\tau]$ into $L^1(\Omega,\mathcal{F}_\tau)$. The Proposition then gives a jointly measurable and t-continuous function $N(t,T,\omega)$ which is a version of

$$N(t,T) = \mathbb{E}_{\mathbb{P}^\tau}\big(P^{-1}(T,\tau) \mid \mathcal{F}_t\big).$$

We then choose our version of $P(t,T)$ to be $P(t,\tau)N(t,T)$. The measurability of $\Sigma_i(t,T)$ and $\tilde{\alpha}(t,T)$ follows from Proposition 8. □

Proof of Theorem 2. Result (v). The process $Z_t = P^{-1}(t,\tau)B_t$ has SDE

$$dZ_t = -Z_t \sum_i \Sigma_i(t,\tau)\, dW_i^\tau(t),$$

so that Z_t can be seen to just be a normalisation of ζ_t. (In fact $Z_t = \zeta_t/P(0,\tau)$.) Thus is (v) equivalent to (A1).

Result (vi) follows from (A1) as an application of the converse of the Cameron-Martin-Girsanov theorem for changing measure. For more details, see IV.38.5(ii) of Rogers and Williams (1987). Condition (A1) follows from (vi) by the C-M-G theorem proper.

Result (vii). It is immediate that SR and (vii) are equivalent, so all that remains is to link (vii) with (A1). To prove that (A1) is sufficient for (vii), we consider ζ_t to be the Radon-Nikodym derivative up to time t of $\mathbb{Q}$ with respect to $\mathbb{P}^\tau$. Then for any X in $L^1(\Omega,\mathcal{F}_T,\mathbb{P}^\tau)$

$$\mathbb{E}_{\mathbb{P}^\tau}(X|\mathcal{F}_t) = \zeta_t \mathbb{E}_{\mathbb{Q}}(\zeta_T^{-1}X|\mathcal{F}_t).$$

In particular, for X equal to $P^{-1}(T,\tau)$, then

$$\frac{P(t,T)}{P(t,\tau)} = \frac{B_t}{P(t,\tau)}\mathbb{E}_{\mathbb{Q}}(B_T^{-1} \mid \mathcal{F}_t),$$

which is the desired result (vii).

Conversely, if $B_t^{-1}P(t,\tau)$ is a $\mathbb{Q}$-martingale, for some $\mathbb{Q}$ equivalent to $\mathbb{P}$, then we can perform similar calculations to show that there is a forward measure $\tilde{\mathbb{P}}^\tau$ under which $P^{-1}(t,\tau)B_t$ is a martingale, and so result (v) holds which is equivalent to (A1). (We might have to go back and pick a different forward measure at result (ii) if the market is incomplete, but that doesn't present any serious problems.) □

Proof of Theorem 3. Result (viii). Our basic approach will be define the forward rates via expectations and show that their integral is a bond price, rather than trying to differentiate the bond prices directly.

By Proposition 6, there is a jointly measurable function $F(t,u)$ which, for each u, is a t-continuous version of the martingale

$$F(t,u) = \mathbb{E}_{\mathbb{Q}}\big(r_u B_u^{-1} \mid \mathcal{F}_t\big).$$

By the first part of Proposition 9 (Stochastic Fubini), we can integrate this with respect to u on the interval $[0,T]$

$$\int_0^T F(t,u)\,du = \mathbb{E}_{\mathbb{Q}}\Big(\textstyle\int_0^T r_u B_u^{-1}\,du \,\Big|\, \mathcal{F}_t\Big) = 1 - \mathbb{E}_{\mathbb{Q}}\big(B_T^{-1} \mid \mathcal{F}_t\big).$$

Thus

$$P(t,T) = B_t\left(1 - \int_0^T F(t,u)\,du\right).$$

So that $P(t,T)$ is absolutely continuous in T and

$$-\frac{\partial}{\partial T}P(t,T) = B_t F(t,T).$$

Clearly, $F(t,t) = r_t B_t^{-1}$, so that $-\frac{\partial}{\partial T}P(t,t)$ is r_t. The forward rate $f(t,T)$ is just $B_t F(t,T)/P(t,T)$. Writing $Z(t,T)$ for $B_t^{-1}P(t,T)$, then $f(t,T) = F(t,T)/Z(t,T)$.

Result (ix). Obviously $F(t,T)$ is a $\mathbb{Q}$-martingale, so that f is a semimartingale. Let $\Lambda_i(t,T)$ be the volatility of $F(t,T)$ with respect to $\tilde{W}_i$, so that

$$d_t F(t,T) = \sum_i \Lambda_i(t,T)\,d\tilde{W}_i(t),$$

$$\text{and} \qquad d_t Z(t,T) = Z(t,T)\sum_i \Sigma_i(t,T)\,d\tilde{W}_i(t).$$

By Proposition 8, we can choose a jointly measurable version of Λ_i, which will allow us to integrate it against T later. It can be deduced that

$$d_t f(t,T) = \sum_i \sigma_i(t,T)\,d\tilde{W}_i(t) - \sum_i \sigma_i(t,T)\Sigma_i(t,T)\,dt,$$

where $\sigma_i(t,T)$ is the $\mathcal{F}$-previsible process

$$\sigma_i(t,T) = Z^{-1}(t,T)\big(\Lambda_i(t,T) - F(t,T)\Sigma_i(t,T)\big).$$

Result (x). By the second part of Proposition 9 (Stochastic Fubini),

$$\int_0^T F(t,u)\,du - \int_0^T F(0,u)\,du = \sum_i \int_0^t \left(\int_0^T \Lambda_i(s,u)\,du\right) d\tilde{W}_i(s).$$

But by the proof of part (viii) the left-hand side above is $Z(0,T)-Z(t,T)$, where $Z(t,T) = B_t^{-1}P(t,T)$. This has SDE $d_tZ(t,T) = Z(t,T)\sum_i \Sigma_i(t,T)\,d\tilde{W}_i(t)$. Hence

$$Z(t,T)\Sigma_i(t,T) = -\int_0^T \Lambda_i(t,u)\,du.$$

Now $Z(t,T)$ is absolutely continuous with derivative $-\frac{\partial}{\partial T}Z(t,T) = F(t,T)$, so $\Sigma_i(t,T)$ is absolutely continuous with derivative $-\frac{\partial}{\partial T}\Sigma_i(t,T) = Z^{-1}(t,T)(\Lambda_i(t,T) - F(t,T)\Sigma_i(t,T))$, which is just $\sigma_i(t,T)$. □

The condition (A2), that $\mathbb{E}_{\mathbb{Q}}(\int_0^\tau |r_u|B_u^{-1}\,du) < \infty$, is required to enable application of the stochastic Fubini theorem. In the Heath, Jarrow and Morton (1992) paper, they use a condition (the last part of their C3) which is equivalent to requiring merely that the integral $\int_0^\tau |r_u|B_u^{-1}\,du$ is finite. It is an open question as to whether this is enough, but it is obvious that (A2) is more than sufficient, as there are some weaker pathwise conditions which can be proved to be sufficient.

Additionally, our models may fail the strict meaning of the HJM paper in another way. In particular, a model under Theorems 1–3 will not necessarily satisfy the second inequality of the HJM condition C2 in Heath, Jarrow and Morton (1992), though it will satisfy their inequality (7) which that condition is used to prove.

5. Two Examples

A market with a non-martingale cash bond

Pick a maturity τ, and let W_t be a $\mathbb{P}$-Brownian motion. (Here $\mathbb{P}$ will actually be the forward measure $\mathbb{P}^\tau$ as well.) Then we shall choose for our $P(t,\tau)$ to follow a Bessel(3) process up to some time $\tau_0 < \tau$. Details of this Bessel process can be found in, for example, VI.3 of Revuz and Yor (1994). (How $P(t,\tau)$ evolves after τ_0 will be immaterial.) Then $\mathbb{P}(t,\tau)$ has SDE

$$d_tP(t,\tau) = P(t,\tau)\big(P^{-1}(t,\tau)\,dW_t + P^{-2}(t,\tau)\,dt\big), \qquad t \leqslant \tau_0.$$

This gives the volatility $\Sigma(t,\tau) = P^{-1}(t,\tau)$ and drift $\tilde{\alpha}(t,\tau) = P^{-2}(t,\tau)$. We can define the rest of the market via result (iv) as

$$P(t,T) = P(t,\tau)\mathbb{E}\big(P^{-1}(T,\tau) \mid \mathcal{F}_t\big),$$

which is well-defined as $P^m(t,\tau)$ has finite expectation for all powers $m > -3$. This model satisfies the BM conditions, and has the r_t process equal to $r_t = \Sigma^2(t,\tau) - \tilde{\alpha}(t,\tau) = 0$. This is the nicest interest rate process we could hope for — it is non-negative, bounded and deterministic. But the discounted bond B_t is not a martingale. As $B_t = 1$, the process $P^{-1}(t,\tau)B_t$ is just the reciprocal of a Bessel(3) which by VI.33 of Rogers and Williams (1987) is only a local martingale and not a full martingale. The condition (A1) fails to hold.

Bond prices as Brownian bridges

We can show that even an SR/HJM-style model can have unusual properties. We will need to recall the notation that the left-limit and right-limit of any function f at x can be denoted by $f(x-)$ and $f(x+)$ respectively, or equivalently by $\lim_{y\uparrow x} f(y)$ and $\lim_{y\downarrow x} f(y)$ respectively.

Returning to the HJM model, for example, it is true that the volatility $\Sigma(t,T)$ is absolutely continuous in T and

$$\Sigma(t,t+) = 0,$$

that does not mean that the limit in the other direction

$$\Sigma(t-,t)$$

either exists or is zero. In other words, there is no mathematical reason (as opposed to economic reasons) that a bond's volatility should get smaller as it approaches maturity. If the volatility stays away from zero, we might expect the bond price to follow some sort of Brownian bridge path, but existing models tend to discount this possibility. The fact that non-vanishing volatility is possible has not been well appreciated in the literature. Hull and White (1993) remark that it would imply unbounded drifts, and don't make it clear that $\Sigma(T,T) = 0$ is insufficient to avoid this. Cheng (1991) does show that an exponential Brownian bridge cannot be a bond price in an arbitrage-free market, but in a setting which places restrictions on the interest rate r_t.

We can construct an arbitrage-free complete bond market in which some of the log bond prices behave as Brownian bridges with constant volatility.

Our measure throughout will be the martingale measure $\mathbb{Q}$. Let a be a positive constant, and fix a date τ. The τ-bond will be made to be a 'log-Brownian bridge', that is, $P(t,\tau) = \exp(X_t)$, where X_t ($0 \leqslant t \leqslant \tau$) is a Brownian bridge from $-a\tau$ to 0. The process X has SDE

$$dX_t = \sigma\, dW_t - \frac{X_t}{\tau - t}\, dt,$$

whose solution simultaneously satisfies the two integral equations:

$$X_t + a\tau = \sigma W_t - \int_0^t \frac{X_s}{\tau - s}\, ds,$$

$$\text{and} \qquad X_t + a\tau = at + \sigma(\tau - t)\int_0^t \frac{dW_s}{\tau - s}.$$

Setting r_t to be the previsible interest-rate process

$$\begin{aligned} r_t &= \tfrac{1}{2}\sigma^2 - \frac{X_t}{\tau - t}, \qquad t < \tau, \\ r_\tau &= 0, \end{aligned}$$

our aim is to define a market via Corollary 4. We can re-express r_t, for $t < \tau$, as

$$r_t = \tfrac{1}{2}\sigma^2 + a - \sigma \int_0^t \frac{dW_s}{\tau - s}.$$

We can use this expression to calculate the variance of r_t. The reason for doing that is to be able to discover the asymptotic size of r_t as t nears τ. In fact $\|r_t\|_2 \sim \sigma(\tau - t)^{-\frac{1}{2}}$, which is t-integrable. Thus the first integral condition of Corollary 4 is satisfied. Then we can integrate r between the limits of t and T to get

$$\int_t^T r_u \, du = \tfrac{1}{2}\sigma^2 (T - t) - \frac{T - t}{\tau - t} X_t - \sigma \int_t^T \frac{T - u}{\tau - u} \, dW_u, \quad t \leqslant T \leqslant \tau.$$

Let Z be the normal random variable $\int_t^T \frac{T-u}{\tau-u} \, dW_u$, which is the last term on the right-hand side above (without the factor of σ). This has variance

$$\operatorname{Var}(Z) = \int_t^T \left(\frac{T - u}{\tau - u} \right)^2 du = (T - t) + (\tau - T) \left(\frac{T - t}{\tau - t} + 2 \log \frac{\tau - T}{\tau - t} \right).$$

In the case where $t = 0$, this variance is bounded by τ, and the mean of $\int_0^T r_s \, ds$ is bounded by $(\frac{1}{2}\sigma^2 + a)\tau$. Hence $\|B_t^{-1}\|_2$ is bounded, and so $\|r_t B_t^{-1}\|_1 \leqslant c(\tau - t)^{-\frac{1}{2}}$, which is t-integrable. So the other integral condition of Corollary 4 is satisfied.

Let us define the bond prices $P(t, T)$ to be $P(t, T) = \mathbb{E}(\exp - \int_t^T r_u \, du | \mathcal{F}_t)$ which gives

$$\begin{aligned} P(t, T) &= \exp \left\{ \frac{T - t}{\tau - t} X_t + \tfrac{1}{2}\sigma^2 (\tau - T) \left(\frac{T - t}{\tau - t} + 2 \log \frac{\tau - T}{\tau - t} \right) \right\}, \quad t \leqslant T < \tau, \\ P(t, \tau) &= \exp(X_t), \qquad t \leqslant \tau. \end{aligned}$$

We note that the price of the τ-bond is a log-Brownian bridge process. Extending the r_t process to the interval $(\tau, 2\tau]$ by taking an independent copy of the distribution of the process r on $[0, \tau]$, then

$$P(t, T) = \exp \left\{ X_t - a(T - \tau) + \tfrac{1}{2}\sigma^2 (2\tau - T) \left(\frac{T - \tau}{\tau} + 2 \log \frac{2\tau - T}{\tau} \right) \right\}, \qquad t \leqslant \tau < T < 2\tau.$$

Taking the SDE of these bond price equations, we have that

$$d_t P(t, T) = P(t, T) \big(\Sigma(t, T) \, dW_t + r_t \, dt \big),$$

where Σ is the non-vanishing volatility function

$$\Sigma(t,T) = \begin{cases} \frac{T-t}{\tau-t}\sigma & t \leqslant T < \tau \\ \sigma & t < \tau \leqslant T \\ 0 & t = \tau \leqslant T. \end{cases}$$

Indeed the Σ tends to zero as maturity decreases, that is $\Sigma(\tau,\tau+) = \Sigma(\tau,\tau) = 0$, but not as time increases, because $\Sigma(\tau-,\tau) = \sigma$.

The forward rates also exist with $f(t,T) = -\frac{\partial}{\partial T}\log P(t,T)$, which is

$$f(t,T) = -\frac{X_t}{\tau - t} + \tfrac{1}{2}\sigma^2 + \sigma^2\left(\frac{T-t}{\tau-t} + \log\frac{\tau - T}{\tau - t}\right), \quad t \leqslant T < \tau$$
$$f(t,T) = a + \tfrac{1}{2}\sigma^2\frac{T}{\tau}, \qquad t \leqslant \tau \leqslant T < 2\tau.$$

Their SDEs are

$$d_t f(t,T) = -\frac{\sigma}{\tau - t}\,dW_t + \sigma^2\frac{T-t}{(\tau-t)^2}\,dt, \quad t \leqslant T < \tau$$
$$d_t f(t,T) = 0, \qquad t \leqslant \tau \leqslant T < 2\tau.$$

This is an HJM-style model with

$$\sigma(t,T) = -\frac{\sigma}{\tau - t}, \quad t \leqslant T < \tau$$
$$\sigma(t,T) = 0, \qquad t \leqslant \tau \leqslant T < 2\tau,$$

We can confirm that the equations of results (ix) and (x) hold.

We notice that the behaviour of the process r_t, which is thought of as the short-term interest rate, is rather erratic in this case. In particular, $\sup_{t<\tau}|r_t|$ is infinite almost surely. We should remember that the bond prices and the cash bond however are quite well-behaved, and we never did posit even the existence of r_t initially.

In this case interest rates also go very negative with $\inf_{t<\tau} r_t = -\infty$. This need not happen though. For instance, if we define X_t to be a driftless bridge with non-vanishing volatility, which has SDE

$$dX_t = \max\left\{\epsilon, \frac{|X_t|}{\sqrt{\tau - t}}\right\}\,dW_t,$$

where ϵ is a positive constant, then $X_\tau = 0$ a.s. Then, with $\sigma_t = \max\{\epsilon, |X_t|/\sqrt{\tau - t}\}$, $P(t,\tau) = \exp(X_t)$ and $r_t = \frac{1}{2}\sigma_t^2$, we can create an SR/HJM market in a similar way, but with positive interest rates, though these too are unbounded.

We can even make some progress towards a general result linking non-vanishing volatilities with unbounded interest rates.

Proposition 10. *In an SR model (or equivalently, a general model satisfying BM and (A1)), for any maturity T satisfying $P(0,T) < 1$, it is impossible that both the volatility be bounded below and the short rate be bounded above.*

Proof of Proposition. (For simplicity, we take a one-factor model, but this makes no difference, as each bond in isolation is equivalent to single-factor model.) For a fixed maturity T, let σ_t be the bond volatility $\Sigma(t,T)$. Then rewriting result (vi) of Theorem 2, under the martingale measure, we have that

$$d_t P(t,T) = P(t,T)\big(\sigma_t \, dW_t + r_t \, dt\big).$$

To try and derive a contradiction, let us suppose that $\sigma_t \geqslant \epsilon$ for some positive ϵ and $|r_t| \leqslant K$ for some constant K, for all $t \leqslant T$. Then we set $\gamma_t = r_t/\sigma_t$, which is absolutely bounded as $|\gamma_t| \leqslant K/\epsilon$. The Cameron-Martin-Girsanov theorem applies to give a measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ under which

$$\tilde{W}_t = W_t + \int_0^t \gamma_s \, ds \quad \text{is } \mathbb{Q}\text{-Brownian motion.}$$

Thus $d_t P(t,T) = P(t,T)\sigma_t \, d\tilde{W}_t$, and so $P(t,T)$ is a local $\mathbb{Q}$-martingale. As it is non-negative it is also a $\mathbb{Q}$-supermartingale in that

$$\mathbb{E}_{\mathbb{Q}}\big(P(t,T) \bigm| \mathcal{F}_s\big) \leqslant P(s,T), \qquad s \leqslant t \leqslant T.$$

Evaluating this inequality at $s = 0$, $t = T$ gives the contradiction $1 \leqslant P(0,T)$. □

Note. This does not mean that a shrinking Σ and an unbounded $|r_t|$ are mutually exclusive. It is possible that both can happen at once. For instance, a market based on a log-bond price X_t, with SDE

$$dX_t = \sigma(T-t)\, dW_t - \frac{X_t}{(T-t)^\alpha}\, dt,$$

for any $\alpha > 2$, has $\Sigma(t,T) \downarrow 0$ as $t \uparrow T$, but $\limsup_{t\uparrow T} |r_t| = \infty$.

6. Conclusions

What we have done in our three principal theorems is to describe a model with as few conditions as possible. In doing so, two things became apparent. Firstly, that (essentially) all models adapted to a finite-dimensional Brownian motion are in fact short-rate/Heath-Jarrow-Morton models, whether or not that was intended. This particularly underlines the generality and rigorous approach of the Heath, Jarrow and Morton paper (1992), and suggests that the model cannot be ignored by model makers, who *de facto* are working within it. At a rarified level, all such interest-rate models are just restrictions of the general HJM model and/or changes of its notation. At a practical level, however, a convenient notation for a sub-model may reveal the wood which the HJM trees obscure.

What we have not been about here is creating new models. The BM framework resembles the philosopher's ladder that we climb up only in order to be able to throw it away. The BM model, the most general we could think of, turns out just to be both the SR and the HJM models. This means not that we should phrase things in terms of the BM model, but rather that it is pointless to look outside SR/HJM for new Brownian based models. Model-makers should focus on restricting the SR/HJM model to sub-models which suit their particular needs.

Secondly, we have seen that many of the 'conditions' of the HJM model can be taken as proved theorems of the generalised model. In particular, the assumptions about the measurability and integrability of the forward rate volatilities and drifts; the existence of the interest rate process r_t and the regularity of the cash bond B_t; and the T-differentiability of the bond prices are not necessary conditions. Instead they follow from the arbitrage-free nature of the market and the two integrability conditions (A1–2) in the preambles to Theorems 2 and 3.

It is tempting and not entirely unjustified to deduce that, as the HJM model is no better than the SR model, it is pointless to work within its notation rather than the simpler framework of the short-rate. In the end, this is a question that others must decide, but one should beware one thing. In the multi-factor setting that we have worked with throughout, it is true that bond and option prices can be written in terms of the short rate. The price at time t of a claim X maturing at time T is

$$\mathbb{E}_{\mathbb{Q}}\Big(\exp\big(-\textstyle\int_t^T r_s\, ds\big)X \;\Big|\; \mathcal{F}_t\Big).$$

We must remember that although the discount factor is a function of the short-rate process, the claim X might not be and the filtration $\mathcal{F}_t$ almost certainly will not be. In the multi-factor setting, it is necessary to keep track of all the factors, and the HJM notation is set up to help with that, whereas the SR notation conceals it.

Further generalisations will come in weakening some of the conditions of the theorem. Most notably, in a market comprising an infinite number of securities we need some version of the results of Delbaen and Schachermayer (1994) giving the equivalence of a no-arbitrage condition and the existence of a martingale measure (BM2). There is also the question of completeness which we have not tried to address here.

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References

CHENG, S.T. (1991). On the Feasibility of Arbitrage-Based Option Pricing when stochastic bond price processes are involved. *J. Econ. Theory.* **53** 185–198.

DELBAEN, F. and SCHACHERMAYER, W. (1994) The fundamental theorem of asset pricing. *Math. Ann.* **300** 463–520.

HARRISON, J.M. and PLISKA, S.R. (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stoch. Procs. and their Applns.* **11** 215–260.

HEATH, D., JARROW, R. and MORTON, A. (1992). Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation. *Econometrica.* **60** 77–105.

HULL, J. and WHITE, A. (1993). One-factor interest-rate models and the valuation of interest-rate derivative securities. *J. Financial and Quantitative Analysis.* **28** 235–254.

IKEDA, N. and WATANABE, S. (1981). *Stochastic Differential Equations and Diffusion Processes.* North-Holland.

REVUZ, D. and YOR, M. (1994). *Continuous Martingales and Brownian Motion.* Springer-Verlag.

ROGERS, L.C.G. (1995). Which model for the term-structure of interest rates should one use ? In *Mathematical Finance* (ed. M.H.A.Davis, D.Duffie *et al*), IMA Volume 65. Springer-Verlag, 93–116.

ROGERS, L.C.G. and WILLIAMS, D. (1987). *Diffusions, Markov Processes and Martingales, Volume 2: Itô Calculus.* Wiley.

ROGERS, L.C.G. and WILLIAMS, D. (1994). *Diffusions, Markov Processes and Martingales, Volume 1: Foundations.* Wiley.